UDC 539.374
Doi: 10.31772/2587-6066-2020-21-2-201-205

For citation: Senashov S. I., Savostyanova I. L., Cherepanova O. N. Elastic-plastic problem in the case of inhomogeneous plasticity under complex shear conditions. *Siberian Journal of Science and Technology.* 2020, Vol. 21, No. 2, P. 201–205. Doi: 10.31772/2587-6066-2020-21-2-201-205

Для цитирования: Сенашов С. И., Савостьянова И. Л., Черепанова О. Н Упругопластическая задача в случае неоднородной пластичности в условиях сложного сдвига // Сибирский журнал науки и технологий. 2020. Т. 21, № 2. С. 201–205. Doi: 10.31772/2587-6066-2020-21-2-201-205

ELASTIC-PLASTIC PROBLEM IN THE CASE OF INHOMOGENEOUS PLASTICITY UNDER COMPLEX SHEAR CONDITIONS

S. I. Senashov[1*], I. L. Savostyanova[1], O. N. Cherepanova[2]

[1] Reshetnev Siberian State University of Science and Technology
31, Krasnoyarskii rabochii prospekt, Krasnoyarsk, 660037, Russian Federation
[2] Siberian Federal University,
79, Svobodniy Av., Krasnoyarsk, 660041, Russian Federation
*E-mail: sen@sibsau.ru

In this research, the authors solved a two-dimensional elastic-plastic problem of the stress state under complex shear conditions in the body weakened by a hole that is bounded by a piecewise smooth contour. The stress state of a complex shear occurs in a cylindrical body of infinite length under the action of loads directed along the cylinder generators and constant along the generators. At the same time, with a sufficiently large load, both elastic and plastic zones appear in the body. As in any problem of this kind, it is necessary to find a previously unknown boundary separating the elastic and plastic zones. Finding such a boundary is not an easy task, but the specificity of elastic-plastic problems of complex shear is that solving such problems is easier than solving similar elastic problems. Apparently, for the first time this fact was noted by G. P. Cherepanov.

A lot of research is devoted to elastic-plastic problems of complex shear in the case of homogeneous and isotropic plasticity. All articles that solve complex shear problems essentially use the representation of stresses and displacements in the elastic zone in a complex form. In this research, the problems of complex shear are solved using conservation laws. It is assumed that the yield strength is a function of the coordinates of the point where the stress state is being studied. It is known that the elastic properties of structural materials can be homogeneous and isotropic, while their yield point and strength are inhomogeneous. This situation is observed, for example, in the case of neutron bombardment of structural materials. This research will examine exactly this situation. The article presents conservation laws for equations describing a complex shear. It was assumed that the components of the conserved current depend on the components of the stress tensor and coordinates. The components of the stress tensor are included in them linearly. The problem of finding the components of the conserved current was reduced to the Cauchy-Riemann system. The solution of this system allowed us to reduce the calculations of the stress tensor components to a curvilinear integral along the contour of the hole and thus find the boundary between the elastic and plastic areas.

Keyword: elastic-plastic problem, inhomogeneous plasticity, complex shear, conservation laws.

УПРУГОПЛАСТИЧЕСКАЯ ЗАДАЧА В СЛУЧАЕ НЕОДНОРОДНОЙ ПЛАСТИЧНОСТИ В УСЛОВИЯХ СЛОЖНОГО СДВИГА

С. И. Сенашов[1*], И. Л. Савостьянова[1], О. Н. Черепанова[2]

[1] Сибирский государственный университет науки и технологий имени академика М. Ф. Решетнева
Российская Федерация, 660037, г. Красноярск, просп. им. газ. «Красноярский рабочий», 31
[2] Сибирский федеральный университет,
Российская Федерация, 660041, г. Красноярск, просп. Свободный, 79
*E-mail: sen@sibsau.ru

В работе решена плоская упругопластическая задача о напряженном состоянии в условиях сложного сдвига в теле, ослабленном отверстием, которое ограниченно кусочно гладким контуром. Напряженное состояние сложного сдвига возникает в цилиндрическом теле бесконечной длины под действием нагрузок, направленных по образующим цилиндра и постоянным вдоль образующих. При этом при достаточно большой нагрузке в теле возникают как упругие, так и пластические зоны. Как и в любой задаче подобного рода возникает необходимость в нахождении заранее неизвестной границы, разделяющей упругую и пластическую зоны. Отыскание такой границы не простая задача, но специфика упругопластических задач о сложном сдвиге состоит в том, *что решение подобных задач проще, чем решение аналогичных упругих задач. По-видимому, впервые этот факт отметил Г. П. Черепанов.*

Упругопластическим задачам о сложном сдвиге в случае однородной и изотропной пластичности посвящена обширная литература. Во всех статьях, в которых решаются задачи о сложном сдвиге, существенно используют представление напряжений и смещений в упругой зоне в комплексном виде. В предлагаемой работе решены задачи о сложном сдвиге с помощью законов сохранения. При этом предполагается, что предел текучести является функцией от координат точки, в которой исследуется напряженное состояние. Известно, что упругие свойства конструкционных материалов могут быть однородными и изотропными, а при этом их предел текучести и прочности – неоднородным. Такая ситуация наблюдается, например, при нейтронной бомбардировке конструкционных материалов. В данной статье будет изучена именно такая ситуация. В статье приведены законы сохранения для уравнений, описывающих сложный сдвиг. При этом предполагалось, что компоненты сохраняющегося тока зависят от компонент тензора напряжений и координат. Компоненты тензора напряжений входят в них линейно. Задача о нахождении компонент сохраняющегося тока свелась к системе Коши–Римана. Решение этой системы позволили свести вычисления компонент тензора напряжений к криволинейному интегралу по контуру отверстия и тем самым найти границу между упругой и пластической областями.

Ключевые слова: упругопластическая задача, неоднородная пластичность сложный сдвиг, законы сохранения.

Introduction. The stress state of a complex shear occurs in a cylindrical body of infinite length under the action of loads directed along the generators of the cylinder and constant along the generators [1]. At the same time, with a sufficiently large load, both elastic and plastic zones appear in the body. As in any problem of this kind, it is necessary to find a previously unknown boundary separating the elastic and plastic zones. Finding such a boundary is not an easy task, but the specificity of elastic-plastic problems about complex shear is that the solution of such problems is easier than the solution of similar elastic problems. Apparently, this fact was first noted by G. P. Cherepanov [1].

A lot of research is devoted to the elastic-plastic problems of complex shear in the case of homogeneous and isotropic plasticity, and its review can be seen, for example, in [1]. In all articles that solve complex shear problems, the representation of stresses and displacements in the elastic zone in a complex form is significantly used. In this research, we solve the problem of complex shear using conservation laws. For the first time, plasticity problems were solved using conservation laws in [2–6]. In articles [3; 7–12], the method of conservation laws was successfully applied to finding elastic-plastic rods and beams. In this article, this technique is used for the first time to solve elastic-plastic problems. And it is assumed that in the plastic area, the yield strength is a function of the coordinates of the point at which the stress state is studied. It is known that the elastic properties of structural materials can be homogeneous and isotropic, and their yield strength and strength – inhomogeneous. This situation is observed, for example, in the case of neutron bombardment of structural materials [13]. This article will study exactly this situation.

1. Basic ratios. The shear and stress fields in this case are as follows [1]

$$\begin{gathered} u = v = 0,\ \ w = w(x, y), \\ \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0, \\ \tau_{xz} = \tau^1(x, y), \tau_{yz} = \tau^2(x, y). \end{gathered} \tag{1}$$

Here u, v, w – shear vector components, σ_x, σ_y, $\sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ – stress tensor components, x, y, z – cartesian coordinates, the z axis is directed parallel to the generator

$$\frac{\partial \tau^1}{\partial x} + \frac{\partial \tau^2}{\partial y} = 0, \text{(equilibrium equation)}; \tag{2}$$

$$\tau^1 = G\frac{\partial w}{\partial x},\ \ \tau^2 = G\frac{\partial w}{\partial y}, \text{(Hooke's law)}. \tag{3}$$

Here G – a constant called the shear modulus.

From (2), (3) follow the ratios in the elastic zone

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 0; \tag{4}$$

$$\frac{\partial \tau^1}{\partial y} = \frac{\partial \tau^2}{\partial x}. \tag{5}$$

From (2) and (5) it follows that τ^1, τ^2 satisfy the Cauchy-Riemann equations

$$F_1 = \frac{\partial \tau^1}{\partial x} + \frac{\partial \tau^2}{\partial y} = 0,\ \ F_2 = \frac{\partial \tau^1}{\partial y} - \frac{\partial \tau^2}{\partial y} = 0. \tag{6}$$

In the plastic area, there is a ratio (2), and also

$$(\tau^1)^2 + (\tau^2)^2 = k^2 \text{ (the yield condition)}; \tag{7}$$

$$\tau^2 \frac{\partial w}{\partial x} = \tau^1 \frac{\partial w}{\partial y} \text{ (Genki equation)}. \tag{8}$$

Here $k(x, y)$ – some smooth function equal to the yield strength at pure shear [13].

At the boundary of elastic and plastic areas, stresses and shears are assumed to be continuous.

2. Conservation laws. The conservation law for the system of equations (6) is the following ratio

$$\frac{\partial A(x, y, \tau^1, \tau^2)}{\partial x} + \frac{\partial B(x, y, \tau^1, \tau^2)}{\partial y} = \omega^1 F_1 + \omega^2 F_2, \tag{9}$$

where $\omega^i = \omega^i(x, y, \tau^1, \tau^2)$ – some functions that are simultaneously not not equal to zero.

Comment. A more general definition of conservation laws and their use in the mechanics of a deformable solid can be found, for example, in [4; 6; 14]. In [15], you can read about the application of the group analysis technique for constructing solutions for the equations of inhomogeneous elasticity theory.

For the purposes set out in the article, a simplified formulation in the form of (9) is quite appropriate.

In the case of (9), the values A and B are called the components of the conserved current.

Let us assume that components A And B have the following form

$$A=\alpha^1\tau^1+\beta^1\tau^2+\gamma^1, B=\alpha^2\tau^1+\beta^2\tau^2+\gamma^2, \quad (10)$$

where $\alpha^i=\alpha^i(x,y), \beta^i=\beta^i(x,y), \gamma^i=\gamma^i(x,y)$ – some smooth functions to be defined. Substituting (10) in (9), as a result we get

$$\alpha^1_x\tau^1+\alpha^1\tau^1_x+\beta^1_x\tau^2+\beta^1\tau^2_x+\gamma^1_x+\alpha^2_y\tau^1+\alpha^2\tau^1_y+\beta^2_y\tau^2+$$
$$+\beta^2\tau^2_y+\gamma^2_y=\omega^1\left(\tau^1_x+\tau^2_y\right)+\omega^2\left(\tau^1_y-\tau^2_x\right)=0, \quad (11)$$

where the index at the bottom means the derivative of the corresponding variable.

From (11) we get

$$\begin{aligned}&\alpha^1=\omega^1, \beta^1=-\omega^2,\ \alpha^2=\omega^2,\ \beta^2=\omega^1,\\&\alpha^1_x+\alpha^2_y=0, \beta^1_x+\beta^2_y=0,\ \gamma^1_x+\gamma^2_y=0.\end{aligned} \quad (12)$$

From (12), excluding ω^i we get

$$\alpha^1_x-\beta^1_y=0, \beta^1_x+\alpha^1_y=0, \gamma^1_x+\gamma^2_y=0. \quad (13)$$

Due to the ratios (12), the components of the conserved current have the form

$$A=\alpha^1\tau^1+\beta^1\tau^2+\gamma^1,\ B=-\beta^1\tau^1+\alpha^1\tau^2+\gamma^2. \quad (14)$$

Since the right part (9) is zero, by Green formula we obtain

$$\begin{aligned}&\iint_S\left(A_x+B_y\right)dxdy=\int_{\partial S}Ady-Bdx=\\&=\int_{\partial S}(\alpha^1\tau^1+\beta^1\tau^2+\gamma^1)dy-\left(-\beta^1\tau^1+\alpha^1\tau^2+\gamma^2\right)dx=0.\end{aligned} \quad (15)$$

Where S – area, ∂S – its piecewise smooth border. All functions included in (15) are assumed to be smooth.

Elastic plastic problem for an arbitrary hole in the case when the plastic area covers the entire hole. Let C – a piecewise smooth contour with a load applied to it

$$l^1\tau^1+l^2\tau^2=\tau_n, |\tau_n|\le k, \quad (16)$$

where (l^1,l^2) – the components of the normal vector to contour C. The contour of the plastic area L completely covers the hole C (see fig. 1).

In this case, in addition to condition (16), the yield condition (7) on the contour C is also satisfied. Thus, there are two conditions on C:

$$l^1\tau^1+l^2\tau^2=\tau_n,\ (\tau^1)^2+(\tau^2)^2=k^2. \quad (17)$$

From conditions (17), we find the stress tensor components on the conour C:

$$\begin{aligned}&\tau^1=\tau_n\pm l^2\sqrt{k^2-\tau_n^2},\\&\tau^2=\tau_n l^2\mp l^1\sqrt{k^2-\tau_n^2}.\end{aligned} \quad (18)$$

Further, for certainty in formulas (18), we will choose the upper sign.

4. The use of conservation laws to find the stress tensor component in the area. Let the point $M(x_m,y_m)$ lie outside of contour C. Let us build a radius ε circle centered at point M. We have $\varepsilon:(x-x_m)^2+(y-y_m)^2=\varepsilon^2$. Let D be a straight line connecting point M with contour C. We get a closed contour consisting of a circle ε, a straight line D, and contour C (fig. 2).

From (15) we get

$$\begin{aligned}&\oint_C Ady-Bdx+\int_{D^+}Ady-Bdx+\\&+\int_{D^-}Ady-Bdx+\int_{\varepsilon}Ady-Bdx=0.\end{aligned} \quad (19)$$

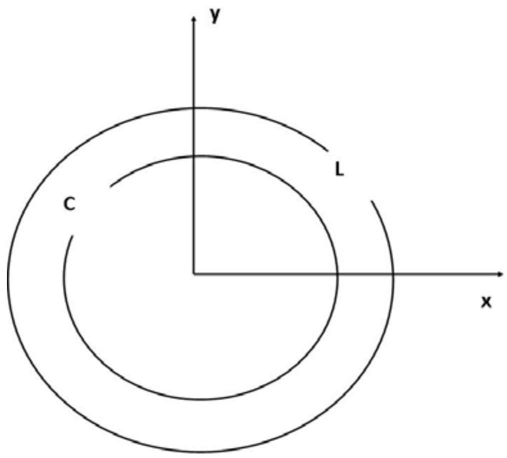

Fig. 1. Elastic-plastic problem for an arbitrary hole in the case when the plastic area covers the entire hole

Рис. 1. Упруго пластическая задача для произвольного отверстия в случае, когда пластическая область охватывает все отверстие

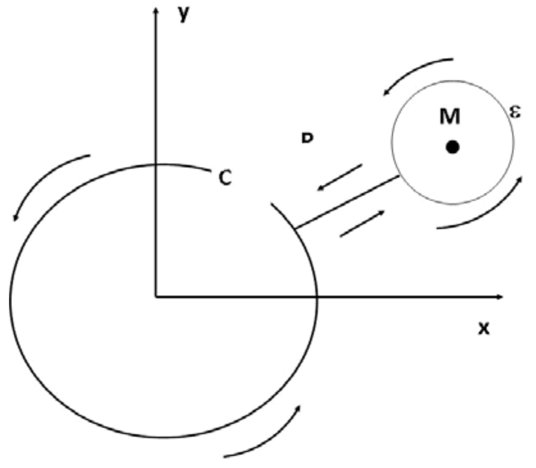

Fig. 2. Closed contour consisting of a circle ε , straight line *D*, and contour *C*

Рис. 2. Замкнутый контур, состоящий из окружности ε , прямой *D* и контура *C*

The sum of the second and third terms in (19) is zero, since the integrals are calculated in different directions. Finally from (19) we have

$$\int_C Ady - Bdx = -\oint_\varepsilon Ady - Bdx. \qquad (20)$$

We transform the right side of equation (20) by introducing parameterization $x = \varepsilon\cos\cos t$, $y = \varepsilon sint$, $0 \le t \le 2\pi$. As a result, we have

$$\oint_\varepsilon Ady - Bdx = \varepsilon\int_0^{2\pi}\left(Acost + Bsint\right)dt. \qquad (21)$$

Let in (15)

$$\alpha^1 = \frac{x}{x^2+y^2},\ \beta^1 = -\frac{y}{x^2+y^2}. \qquad (22)$$

Then from (21) we get

$$\int_\varepsilon A_1dy - B_1dx = \varepsilon\int_0^{2\pi}\left(A_1cost + B_1sint\right)dt = \\ = \int_0^{2\pi}\tau^1 dt = 2\pi\tau^1\left(x_m, y_m\right). \qquad (23)$$

The last equality in (23) is obtained using the theorem on the mean at ε tending to zero.

Let in (15)

$$\alpha^1 = \frac{y}{x^2+y^2},\ \beta^1 = \frac{x}{x^2+y^2}. \qquad (24)$$

Then from (21) we get

$$\oint_\varepsilon A_2dy - B_2dx = \varepsilon\int_0^{2\pi}\left(A_2cost + B_2sint\right)dt = \\ = \int_0^{2\pi}\tau^2 dt = 2\pi\tau^2\left(x_m, y_m\right). \qquad (25)$$

The last equality in (25) is obtained using the theorem on the mean at ε tending to zero.

The last equality in (25) is obtained using the theorem on the mean at ε tending to zero.

From the formula (20), as well as from (23) and (25), we get

$$\oint_C A_1dy - B_1dx = -2\pi\tau^1\left(x_m, y_m\right), \\ \oint_C A_2dy - B_2dx = -2\pi\tau^2\left(x_m, y_m\right). \qquad (26)$$

Formulas (26) make it possible to find the components of the stress tensor at any point outside the contour *C*. This allows to set the boundary between the elastic and plastic areass. If the condition of plasticity $(\tau^1)^2 + (\tau^2)^2 = k^2$ is satisfied at a point x_m, y_m, then this point belongs to the plastic area, when the condition $(\tau^1)^2 + (\tau^2)^2 < k^2$ is satisfied , to the elastic.

Comment. The above formulas allow us to solve elastic-plastic problems, even if the plastic contour does not completely cover the contour *C,* provided that the plasticity condition is satisfied on the contour *C* (7).

References

1. Annin B. D., CHerepanov G. P. *Uprugo plasticheskaya zadacha*. [Elastic plastic task] Novosibirsk, Nauka Publ., 1983, 239 p.

2. Senashov S. I. [On the laws of conservation of plasticity equations]. *Dokl. AN SSSR*. 1991, Vol. 320, No. 3, P. 606–608 (In Russ.).

3. Senashov S. I., Filyushina E. V. *Uprugoplasticheskie zadachi dlya ortotropnyh sred*. [Elastic-plastic problems for orthotropic environments]. Krasnoyarsk, SibGU im. M. F. Reshetneva Publ., 2017, 116 p.

4. Kiryakov P. P., Senashov S. I., Yahno A. N. *Prilozhenie simmetrij i zakonov sohraneniya k resheniyu differencial'nyh uravneniy.* [Application of symmetries and conservation laws to the solution of differential equations]. Novosibirsk, Nauka Publ., 2001, 192 p.

5. Senashov S. I., Gomonova O. V., Yahno A. N. *Matematicheskie voprosy dvumernyh uravnenij ideal'noj plastichnosti.* [Mathematical problems of two-dimensional equations of ideal plasticity] Sib. gos. aerokosmich. un-t. Krasnoyarsk, 2012. 139 p.

6. Ivlev D. D. et al. *Predel'noe sostoyanie deformirovannyh tel i gornyh porod* [Limit state of deformed bodies and rocks]. Moscow, FIZMTLIT Publ., 2008.

7. Senashov S. I., Filyushina E. V. [Analytical solution of the problem of the load wave in an elastic-plastic rod]. *Dinamika sploshn. sredy*. 2012, No. 127.

8. Senashov S. I., Filyushina E. V., Gomonova O. V. [Building elastic-plastic boundaries using conservation laws]. *Vestnik SibGAU*. 2015, Vol. 16, No. 2, P. 343–359 (In Russ.).

9. Senashov S. I., Kondrin A. V. [Development of an information system for finding the elastic-plastic boundary of rolling profile rods]. *Vestnik SibGAU*. 2014, No. 4(56), P. 119–125 (In Russ.).

10. Senashov S. I., Cherepanova O. N., Kondrin A. V. [About elastic-plastic torsion of a rod]. *Vestnik SibGAU*. 201, No. 3(49), P. 100–103 (In Russ.).

11. Senashov S. I., Cherepanova O. N., Kondrin A. V. Elastoplastic Torsion of a Rod with MultiplyConnected Cross-Section. *J. Siberian Federal Univ. Math. & Physics*. 2015, No. 7(1), P. 343–351.

12. Senashov S. I., Cherepanova O. N., Kondrin A. V. On Elastoplastic Bending of Beam. J. *Siberian Federal Univ. Math. & Physics*. 2014, No. 7(2), P. 203–208.

13. Ol'shak V., Mruz Z., Pezhina P. *Neodnorodnaya teoriya plastichnosti* [Heterogeneous theory of plasticity] Moscow, Mir Publ., 1964, 156 p.

14. Senashov S. I., Vinogradov A. M. Symmetries and conservation laws of 2-dimensional ideal plasticity. *Proc. EdinburgMath. Soc.* 1988, Vol. 3(2), P. 415–439.

15. Annin B. D., Bytev V. O., Senashov S. I. *Gruppovye svojstva uravnenij uprugosti i plastichnosti* [Group properties of elasticity and plasticity equations]. Novosibirsk, Nauka Publ., 1985, 143 p.

Библиографические ссылки

1. Аннин Б. Д., Черепанов Г. П. Упруго пластическая задача. Новосибирск : Наука, 1983. 239 с.

2. Сенашов С. И. О законах сохранения уравнений пластичности // Докл. АН СССР. 1991. Т. 320, № 3. С. 606–608.

3. Сенашов С. И., Филюшина Е. В. Упругопластические задачи для ортотропных сред / СибГУ им. М. Ф. Решетнева. Красноярск, 2017. 116 с.

4. Киряков П. П., Сенашов С. И., Яхно А. Н. Приложение симметрий и законов сохранения к решению дифференциальных уравнений. Новосибирск : Наука, 2001. 192 с.

5. Сенашов С. И., Гомонова О. В., Яхно А. Н. Математические вопросы двумерных уравнений идеальной пластичности / Сиб. гос. аэрокосмич. ун-т. Красноярск, 2012. 139 с.

6. Предельное состояние деформированных тел и горных пород / Д. Д. Ивлев и др. М. : Физмтлит, 2008.

7. Сенашов С. И., Филюшина Е. В. Аналитическое решение задачи о волне нагрузки в упругопластическом стержне // Динамка сплошн. среды. 2012. Вып. 127.

8. Сенашов С. И., Филюшина Е. В., Гомонова О. В. Построение упругопластических границ с помощью законов сохранения // Вестник СибГАУ. 2015. Т. 16, № 2. С. 343–359.

9. Сенашов С. И., Кондрин А. В. Разработка информационной системы для нахождения упругопластической границы стержней прокатного профиля // Вестник СибГАУ. 2014. № 4(56). С. 119–125.

10. Сенашов С. И., Черепанова О. Н., Кондрин А. В. Об упругопластическом кручении стержня // Вестник СибГАУ. 2013. № 3(49). С. 100–103.

11. Сенашов С. И., Черепанова О. Н., Кондрин А. В. Elastoplastic Torsion of a Rod with MultiplyConnected Cross-Section // J. Siberian Federal Univ. Math. & Physics. 2015. No. 7(1). P. 343–51.

12. Сенашов С. И., Черепанова О. Н., Кондрин А. В. On Elastoplastic Bending of Beam // J. Siberian Federal Univ. Math. & Physics. 2014. No. 7(2). P. 203–208.

13. Ольшак В., Мруз З., Пежина П. Неоднородная теория пластичности. М. : Мир, 1964. 156 с.

14. Senashov S. I., Vinogradov A. M. Symmetries and conservation laws of 2-dimensional ideal plasticity // Proc. EdinburgMath. Soc. 1988. Vol. 3(2). P. 415–439.

15. Аннин Б. Д., Бытев В. О., Сенвшов С. И. Групповые свойства уравнений упругости и пластичности. Новосибирск : Наука. Сиб. отделение, 1985. Бытев 143 с.

Senashov Sergei Ivanovich – D. Sc., Professor, Head of the Department of IES; Reshetnev Siberian State University of Science and Technology. E-mail: sen@sibsau.ru.

Savostyanova Irina Leonidovna – Cand. Sc., Associate Professor of the Department of IES, Reshetnev Siberian State University of Science and Technology. E-mail: savostyanova@sibsau.ru.

Cherepanova Olga Nikolaevna – Cand. Sc., associate Professor, acting Director of the Institute of Mathematics and Fundamental Informatics; Siberian Federal University. E-mail: cheronik@mail.ru.

Сенашов Сергей Иванович – доктор физико-математических наук, профессор, заведующий кафедрой ИЭС; Сибирский государственный университет науки и технологий имени академика М. Ф. Решетнева. E-mail: sen@sibsau.ru.

Савостьянова Ирина Леонидовна – кандидат педагогических наук, доцент кафедры ИЭС; Сибирский государственный университет науки и технологий имени академика М. Ф. Решетнева. E-mail: savostyanova@sibsau.ru.

Черепанова Ольга Николаевна – кандидат физико-математических наук, доцент, исполняющая обязанности директора Института математики и фундаментальной информатики; Сибирский федеральный университет. E-mail: cheronik@mail.ru.