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THE METHOD OF EQUIVALENT STRENGTH CONDITIONS IN CALCULATING COMPOSITE STRUCTURES WITH A REGULAR STRUCTURE USING MULTIGRID FINITE ELEMENTS

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Plates, beams and shells with non-uniform and micro-inhomogeneities regular structure are widely used in aviation and rocket and space technology. At the preliminary design stage, it is initially important to know whether the design safety factor meets the specified strength conditions. To determine the margin factor, it is necessary to solve the elasticity problem for the designed structure by the finite element method (FEM), taking into account its inhomogeneous structure, which requires large computer resources. In this paper, we propose a method of equivalent strength conditions (MESC) for calculating the static strength of elastic structures with a inhomogeneous regular structure. The proposed method is reduced to the calculation of the strength of isotropic homogeneous bodies using equivalent strength conditions. The MESC is based on the following statement. For any composite body V_0*, there exists such an isotropic homogeneous body* V^b *and such a number* p *(equivalence coefficient) that if the body* V^b *stock coefficient satisfies* n_b^0 *the equivalent strength conditions* $pn_1 \le n_b^0 \le pn_2$*, then the body* V_0 *stock coefficient satisfies* n_0 *the given strength conditions* $n_1 \le n_0 \le n_2$*, and Vice versa,* n_1*,* n_2*– given, the coefficients* n_b^0*,* n_0*, meet the exact solutions of elasticity problems constructed for bodies* V_0*,* V^b*. The method under consideration is reduced to FEM strength calculation of isotropic homogeneous bodies, which is the easiest to implement and requires less computer memory than a similar calculation of composite bodies taking into account their inhomogeneous structure. The procedure for determining the equivalence coefficients for a number of composite plates, beams and shells of rotation is described. High-precision multigrid finite elements generating discrete models of small dimension and solutions with small error are used in the construction of elastic solutions according to FEM for isotropic homogeneous bodies. The adjusted equivalent strength conditions are of the form* $pn_1(1+\varepsilon_1) \le n_b \le pn_2(1-\varepsilon_2)$*, where* n_b *is the body* V^b *reserve coefficient and the values* ε_1*,* ε_2 *correspond to the approximate solution constructed for the body* V^b*. Implementation of FEM for multigrid discrete models requires several* $10^3 \div 10^6$ *times less computer memory than for basic models. The calculation of the strength of a beam with a micro-homogeneous regular structure with the help of MESC is given.*

Keywords: elasticity, composites, equivalent strength conditions, multigrid finite elements, plates, beams, shells.

МЕТОД ЭКВИВАЛЕНТНЫХ УСЛОВИЙ ПРОЧНОСТИ В РАСЧЕТАХ КОМПОЗИТНЫХ КОНСТРУКЦИЙ РЕГУЛЯРНОЙ СТРУКТУРЫ С ПРИМЕНЕНИЕМ МНОГОСЕТОЧНЫХ КОНЕЧНЫХ ЭЛЕМЕНТОВ

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Пластины, балки и оболочки с неоднородной и микронеоднородной регулярной структурой широко применяются в авиационной и ракетно-космической технике. На этапе эскизного проектирования первоначально важно знать, удовлетворяет ли коэффициент запаса конструкции заданным условиям прочности. Для определения коэффициента запаса необходимо решить по методу конечных элементов (МКЭ) задачу упругости для проектируемой конструкции с учетом ее неоднородной структуры, что требует больших ресурсов ЭВМ.
В данной работе предложен метод эквивалентных условий прочности (МЭУП) для расчета на прочность упругих конструкций с неоднородной регулярной структурой. Предлагаемый метод сводится к расчету на прочность изотропных однородных тел с применением эквивалентных условий прочности. В основе МЭУП лежит следующее утверждение. Для всякого композитного тела V_0 *существует такое изотропное однородное тело* V^b *и такое число* p *(коэффициент эквивалентности), что если коэффициент запаса* n_b^0 *тела* V^b *удовлетворяет эквивалентным условиям прочности* $pn_1 \le n_b^0 \le pn_2$*, то коэффициент запаса* n_0 *тела* V_0 *удовлетворяет заданным условиям прочности* $n_1 \le n_0 \le n_2$*, и наоборот,* n_1*,* n_2 *- заданы, коэффициенты* n_b^0*,* n_0 *отвечают точным решениям задач упругости, построенных для тел* V^b *и* V_0*. Расчет на прочность по МКЭ изотропных однородных тел наиболее простой в реализации и требует меньше памяти ЭВМ, чем аналогичный расчет тел с учетом их неоднородной структуры. Изложена процедура определения коэффициента эквивалентности* p *с помощью МКЭ. При построении решений по МКЭ для изотропных однородных тел применяются многосеточные конечные элементы, порождающие модели малой размерности и решения с малой погрешностью. Скорректированные эквивалентные условия прочности имеют вид* $pn_1(1+\varepsilon_1) \le n_b \le pn_2(1-\varepsilon_2)$*, где* n_b *– коэффициент запаса тела* V^b *и величины* ε_1*,* ε_2 *отвечают приближенному решению. Реализация МКЭ для многосеточных дискретных моделей требует в* $10^3 \div 10^6$ *раз меньше объема памяти ЭВМ, чем для базовых. Приведен расчет на прочность балки с микронеоднородной регулярной структурой с помощью МЭУП.*

Ключевые слова: упругость, композиты, эквивалентные условия прочности, многосеточные конечные элементы, пластины, балки, оболочки.

Introduction. The calculation of structural strength is one of the most important at the stage of a schematic design [1], which represents the technical and economic assessment of a design project. As a rule, structural strength calculation is performed according to margins of safety [1-3]. According to the calculation, for the safety factor n_0 of the designed structure V_0, the specified strength conditions have the following form

$$n_1 \le n_0 \le n_2, \qquad (1)$$

where n_1, n_2 are given, $n_1 > 1$.

At the stage of schematic design, it is primarily important for the designer to know whether the safety factor n_0 of the designed structure V_0 satisfies the specified strength conditions or not (1).

If the safety factor n_0 satisfies the specified strength conditions, it is considered that the structure V_0 does not fracture under given operational conditions. It should be noted that in this case it is not necessary to examine in detail the stress-strain state of the structure V_0. The calculation of the strength of the structure V_0 reduces to finding its safety factor n_0 and to the strength test (1) for the factor n_0. The safety factor n_0 is defined according to the formula $n_0 = \sigma_T / \sigma_0$ [1–3], where σ_T – limit stress of the structure V_0 (yield stress [3]), σ_0 – maximum equivalent stress of the structure V_0. It should be noted that the safety factor n_0 corresponds to the exact solution of the elasticity problem formulated for the structure. If the maximum equivalent structural stresses are determined approximately, in this case the corrected strength conditions are used [4]. In the analysis of the stress-strain state of composite structures, the finite element method (FEM) is widely used [5-8]. The basic discrete models of structures with an inhomogeneous and microinhomogeneous structure, which consist of the first-order finite element (FE) and take into account their structures within the framework of the microapproach [9], have a very high dimension, which creates difficulties in the implementation of FEM on a computer. For such models, the multigrid finite element method (MFEM) [10–12] is effectively used, in which the multigrid finite element is used [13, 14]. It should be noted that FEM is a special case of multigrid finite element method, and if in solving boundary value problems in FEM multigrid finite elements are used, in this case, multigrid finite element method is implemented.

For practice, it is important to know the error of approximate solution that is used in the calculations. The error of approximate solutions can be estimated when they differ insignificantly from each other and at the same time form a sequence of solutions that quickly converges to the exact solution. When constructing such sequences, the splitting of the initial partition of the body area into FEs is applied. The splitting procedures used for partitions, which are built for a heterogeneous and microinhomogeneous (fibrous) structure, are complex and difficult to implement. Since the fibers have a small thickness, splitting of such partitions leads to a dramatic increase in the dimensions of discrete models. The implementation of FEM for such models requires large computer resources. In addition, certain restrictions are imposed on the law of splitting, due to the fact that at each step of splitting the partitions, it is necessary to take into account the microinhomogeneous structure by microapproach. As known, the splitting procedure used for discrete models of homogeneous isotropic bodies is the simplest to implement and requires less computer memory than for the bodies with an inhomogeneous and micro-inhomogeneous structure (taking into account their structure).

In this paper, the method of equivalent strength conditions is proposed for the static strength analysis of a linearly elastic structure V_0 with a heterogeneous (microinhomogeneous) regular structure consisting of plastic materials. For simplicity, we believe that the body V_0 has a fibrous structure. It is shown that the strength calculation of the composite body V_0 is reduced to the strength calculation (using FEM) of the isotropic homogeneous body V^b. The bodies V_0, V^b have the same shape, size, fastening and loading. The elasticity moduli of the body V^b and the fiber coincide. In the calculations, adjusted equivalent strength conditions of the form are used.

$$pn_1(1+\varepsilon_1) \le n_b \le pn_2(1-\varepsilon_2), \qquad (2)$$

where $\varepsilon_1 = 1/(1-\delta_p) - 1$, $\varepsilon_2 = 1 - 1/(1+\delta_p)$, p – equivalence number, n_b – safety factor of the body V^b, $n_b = \sigma_T / \sigma^b$, σ_T – yield stress of the fiber, σ^b – maximum equivalent stress of the body V^b (being determined with the help of FEM), δ_p – error for the stress σ^b, $0 \le \delta_p < 1$.

It is shown that if the safety factor n_b of the isotropic homogeneous body V^b satisfies the adjusted equivalent strength conditions (2), the safety factor n_0 of the composite body V_0 (corresponding to the exact solution to the elasticity problem) satisfies the given strength conditions (1). Thus, the implementation of the proposed method is reduced to constructing the adjusted equivalent strength conditions (2) and finding the safety factor n_b of the body V^b, i. e., determining the equivalence coefficient p and finding the maximum equivalent stress σ^b for the body V^b using FEM (with the error δ_p). The equivalence coefficient p is found using the formula $p = \sigma_0 / \sigma^b$, where σ_0 is the maximum equivalent stress of the body V_0. Stresses σ_0 for the composite body V_0 and σ^b for an isotropic homogeneous body are determined by FEM (using multigrid finite elements). To estimate the error of approximate solutions, sequences of solutions (obtained using FEM) are used that quickly converge to the accurate ones.

The advantages of the method of equivalent strength conditions are the following. In the calculations, we use isotropic homogeneous structures that have the same shapes and sizes, fastening and loading, as composite structures. When analyzing the strain-stress state of isotropic homogeneous structures according to FEM, multigrid finite elements are used, which allow us to construct sequences of solutions that are fast converging to exact ones. It allows to determine the error for the obtained approximate solutions. Multigrid finite elements for isotropic homogeneous structures generate discrete models of small dimension and approximate solutions with a small error. The implementation of FEM for multigrid discrete models requires $10^3 \div 10^6$ times less computer memory than for basic ones. When implementing multigrid finite elements, splitting procedures for discrete models of composite structures are not used. An example of calculating the strength of a beam with a microinhomogeneous regular fibrous structure using multigrid finite elements is given.

Fundamental principles for the structures. The paper considers three-dimensional structures (bodies) for which the following conditions are satisfied.

Principle 1. Three-dimensional linearly elastic isotropic homogeneous and composite bodies (structures) are considered in Cartesian coordinate systems. These structures consist of ductile materials, have smooth boundaries, static loading, and the same operating conditions. Body loading functions are smooth functions. The bodies have boundaries that do not degenerate into points. The composite bodies consist of heterogeneous modules of isotropic homogeneous bodies, the connections between which are ideal, i.e., at the common boundaries of heterogeneous modules of homogeneous bodies, the functions of displacements and stresses are continuous. The displacements, deformations, and stresses of the multimodular bodies correspond to the Cauchy relations and the Hooke law of the three-dimensional linear problem of the theory of elasticity [15]. Equivalent stresses for elastic bodies are determined by the 4th theory of strength [1].

Equivalent strength conditions and equivalent structural strengths expressed in terms of safety factors. Let us suppose that two elastic structures V_1 and V_2 have the same shape, geometric dimensions, fastenings, and static loads, but differ in elastic moduli. Let the safety conditions be set for the safety factors n_1, n_2 of the structures V_1, V_2 respectively

$$n_a^1 \le n_1 \le n_b^1, \tag{3}$$

$$n_a^2 \le n_2 \le n_b^2, \tag{4}$$

where $n_a^1, n_a^2 > 1$; $n_a^1, n_a^2, n_b^1, n_b^2$ are given; the factor n_1 (n_2) corresponds to the exact solution of the elasticity problem posed for the structure V_1 (V_2).

Let us introduce the following definitions for the structures V_1, V_2.

Definition 1. If the fulfillment of conditions (4) for the factor n_2 implies the fulfillment of conditions (3) for the factor n_1 and vice versa: if the fulfillment of conditions (3) for the factor n_1 implies the fulfillment of conditions (4) for the factor n_2, then the strength conditions (3), (4) will be called equivalent strength conditions, respectively, for the structures V_1, V_2.

Definition 2. Let the structures V_1, V_2, for which the conditions (3), (4) are equivalent strength conditions respectively, not be destroyed under the same operating conditions. Then we call the structures V_1, V_2 equivalent in strength.

In practice, the strength equivalence of the structures V_1, V_2 means that instead of the stressed structure V_1, the structure V_2 can be used, and vice versa. It should be noted that, of the two structures equivalent in strength, it is appropriate to use a design that is more technological in manufacturing, meets specified technical requirements and requires less financial costs for manufacturing and operation.

The theorem on the existence of equivalent strength conditions. Let us consider the theorem that proves the existence of equivalent strength conditions for elastic composite structures (bodies).

Theorem 1. Let a predetermined static surface force $\mathbf{q}$ act on the three-dimensional linearly elastic composite body V_0 (located in a Cartesian coordinate system $Oxyz$), i.e., forces acting on the unsecured part of the boundary S_q of the body V_0 and volume forces $\mathbf{p}$, where $\mathbf{q} = \{q_x, q_y, q_z\}^T$, $\mathbf{p} = \{p_x, p_y, p_z\}^T$: q_x, q_y, q_z, p_x, p_y, p_z - smooth function of the coordinates x, y, z.

At the boundary S_u, the body V_0 is rigidly fixed, i.e., at S_u: $u = v = w = 0$, $S_0 = S_u + S_q$, S_0 - smooth boundary of the body V_0. The body V_0 consists of V_i components, i.e., plastic, multimodular isotropic homogeneous bodies V_i, where $i = 1,..., N$, N is the total number of the bodies V_i of the body V_0. Let the maximum equivalent stress of the composite body V_0 arise in the body V_α, $1 \le \alpha \le N$. Let the following strength conditions be given for the safety factor n_0 of the composite body V_0 (which corresponds to the exact solution to the elasticity problem for the body V_0)

$$n_1 \le n_0 \le n_2, \tag{5}$$

where n_1, n_2 are given, $n_1 > 1$.

In this case we deal with such a three-dimensional elastic isotropic homogeneous body V^b and such numbers n_1^p, n_2^p that if safety factor n_b^0 of the body V^b (corresponding to the exact solution to the elasticity problem for the body) satisfies equivalent strength conditions of the form

$$n_1^p \le n_b^0 \le n_2^p, \tag{6}$$

the safety factor n_0 of the composite body V_0 satisfies strength conditions (5), and vice versa. If the safely factor n_0 of the composite body V_0 satisfies the conditions (5), the safety factor n_b^0 of the isotropic homogeneous body V^b satisfies the conditions (6), and between the safety factors n_0, n_b^0 there is a mutual one-to-one association.

Proof. Let a homogeneous isotropic body V^b and a composite body V_0 have the same shape, dimensions, fastening and loading, but differ in elasticity moduli. Let the elastic moduli of a body V^b be equal to the elastic moduli of a body V_α of a composite body V_0, $1 \le \alpha \le N$. We find safety factors using the formulae [1–3]

$$n_0 = \sigma_T / \sigma_0, \tag{7}$$

$$n_b^0 = \sigma_T / \sigma_b^0, \tag{8}$$

where σ_T – yield point of the body V_α [3]; σ_0, σ_b^0 – maximum equivalent stresses arising in the bodies V_0, V^b respectively and corresponding to the exact solutions to the elasticity problems.

Let the safety factor n_0 satisfy strength conditions (5). Then plugging (7) in (5) we get an inequation

$$n_1 \le \frac{\sigma_T}{\sigma_0} \le n_2. \tag{9}$$

There is a number p (equivalence number) that

$$p = \sigma_0 / \sigma_b^0. \tag{10}$$

Taking into consideration (10) in (9) we have

$$pn_1 \le \frac{\sigma_T}{\sigma_b^0} \le pn_2. \tag{11}$$

Using (8) in (11) we get

$$pn_1 \le n_b^0 \le pn_2. \tag{12}$$

There are numbers n_1^p, n_2^p that

$$n_1^p = pn_1, \quad n_2^p = pn_2. \tag{13}$$

Plugging (13) in (12) we get that for the factor n_b^0 the conditions are met (6). Thus, there are such numbers n_1^p, n_2^p that the safety factor n_b^0 of the isotropic homogeneous body V^b satisfies the conditions (6). Let the safety factor n_b^0 of the body V^b satisfy strength conditions (6). Plugging (8) in (6) and taking into account (10), (13), we get

$$pn_1 \le \frac{p\sigma_T}{\sigma_0} \le pn_2.$$

Whence taking into account (7), the fulfillment of the strength conditions (5) for the safety factor n_0 of the composite body V_0 follows (5). Thus, it is shown that each factor $n_b \in (n_1^p, n_2^p)$ corresponds to a single factor $n_0 \in (n_1, n_2)$ found using the formula (7), and vice versa, a single coefficient $n_b^0 \in (n_1^p, n_2^p)$ (corresponding to the formula (8)) corresponds to each factor $n_0 \in (n_1, n_2)$. Let us consider the limiting cases. Let $n_b^0 = n_1^p$. Using relations (8), (13), (10) in the last equality, we get $p\sigma_T / \sigma_0 = pn_1$. Whence, taking into account (7), $n_0 = n_1$ follows. Similarly, we can show that if $n_b^0 = n_2^p$, then $n_0 = n_2$. Let $n_0 = n_1$. Using (7), (10) in the last equality, we get $\sigma_T / \sigma_b^0 = pn_1$. Whence, taking into account (8), (13), $n_b^0 = n_1^p$ follows. Similarly, we can show that if $n_0 = n_2$, then $n_b^0 = n_2^p$. Thus, between the safety factors n_0 and n_b^0 there is one-to-one relationship. The theorem 1 is proved.

Equivalent strength conditions (6) can be represented through the equivalence number p in the form $pn_1 \le n_b^0 \le pn_2$, the construction of which is reduced to finding the number p.

It should be noted that the conditions (5), (6) are equivalent strength conditions for bodies V_0, V^b (structures), respectively, see the Definition 1. It is believed that if n_0 satisfies the specified strength conditions (5), then the structure V_0 does not collapse during operation. Let the structure V^b not collapse during operation. Then the structures V_0, V^b are equivalent in strength (see the Definition 2).

Thus, the existence of equivalent strength conditions for composite bodies (structures) having any structure, shape, any size, static loading and fastening that meet the above stated principle 1 and the conditions of the Theorem 1 is proved. It should be noted that for any composite V_0 it is generally possible to construct an isotropic homogeneous structure V^b, i. e., it is always possible to set the shape, dimensions, loading, fastening and elastic moduli according to certain rules for the structure V^b. However, in the general case, equivalent strength conditions for an isotropic homogeneous structure V^b can only be constructed for the given forces $\mathbf{q}$, $\mathbf{p}$, which is impractical. This is due to the fact that the stresses σ_0, σ_b^0 and p correspond to the given loading of the structures. See the formulae (10), (13).

<u>*Comment 1.*</u> Let the value p and the maximum equivalent stress σ_b^0 of the structure V^b be found. Then, for the structure V_0 using formula (10), we determine the maximum equivalent stress σ_0, i.e., $\sigma_0 = p\sigma_b^0$, and then, using the formula (7), we calculate the safety factor n_0, i. e. $n_0 = \sigma_T /(p\sigma_b^0)$, that is important to know when designing a structure.

Corrected strength conditions taking into account the error of stresses. In the general case (e.g., for the bodies having a complex shape) it is very difficult to construct analytical solutions to the three-dimensional problem of the theory of elasticity. However, using FEM [5–8] and FEM [10–12], it is possible to construct approximate solutions to the problems of the theory of elasticity with a given error for stresses. It should be noted that when designing a number of structures (e.g. structures of minimum weight), the violation of the specified strength conditions (5), i. e., equivalent strength conditions (6), is unacceptable. Equivalent strength conditions (6) do not take into account the error of approximate solutions, which creates difficulties in their implementation.

Let the following strength conditions be set for the safety factor of the elastic structure V^b

$$n_1^p \le n_b^0 \le n_2^p \,, \tag{14}$$

where n_1^p, n_2^p are given; n_b^0 – safety factor of the structure V^b, corresponding to the exact solution to the three-dimensional elasticity problem (formulated for this structure).

In the Theorem 2, the corrected strength conditions are formulated taking into account the error of approximate solutions. For convenience and continuity of presentation, in the Theorem 2 we use the notation introduced in the Theorem 1.

<u>***Theorem 2.***</u> Let the strength conditions (14) be given for an elastic structure V^b and the maximum equivalent stress σ_b be defined corresponding to the approximate solution to the elasticity problem. Let

$$|\delta| \le \delta_p < C_p = \frac{\Delta n}{n_1^p + n_2^p} \,, \tag{15}$$

where $\Delta n = | n_2^p - n_1^p |$, n_1^p, n_2^p are given; δ – relative error for voltage σ_b, i. e.

$$\delta = \frac{\sigma_b - \sigma_b^0}{\sigma_b^0} \,, \tag{16}$$

where σ_b^0 is the maximum equivalent stress of the structure V^b corresponding to the exact solution to the problem of elasticity, stresses σ_b^0, σ_b are determined by the 4th theory of strength, δ_p is the estimate for the error δ.

Let the structure safety factor n_b corresponding to the approximate solution satisfy the adjusted strength conditions of the form

$$\frac{n_1^p}{1-\delta_p} \le n_b \le \frac{n_2^p}{1+\delta_p} \,. \tag{17}$$

where $n_b = \sigma_T / \sigma_b$, σ_T – yield stress.

Then the structure safety factor n_b^0 corresponding to the approximate solution satisfies the adjusted strength conditions (14), where $n_b^0 = \sigma_T / \sigma_b^0$.

Proof. From (16) it follows $\sigma_b = (1+\delta)\,\sigma_b^0$. From here we have

$$n_b^0 = (1+\delta) n_b . \tag{18}$$

Note that in (15) $C_p < 1$. Let δ_0 have the value that $\delta_0 = |\delta|$. Then by reason of (15) we have the following formula

$$0 \le \delta_0 = |\delta| \le \delta_p < 1 . \tag{19}$$

Taking into account (18) in sequence $\delta = -\delta_0$, $\delta = \delta_0$, we introduce the factors

$$n_1^r = (1-\delta_0) n_b, \quad n_2^r = (1+\delta_0) n_b, \tag{20}$$

Then by the reason of (18), (20) we get

$$n_b^0 = n_1^r \quad \text{or} \quad n_b^0 = n_2^r . \tag{21}$$

Let us introduce the factors n_1^d, n_2^d according to the formulae

$$n_1^d = (1-\delta_p) n_b, \quad n_2^d = (1+\delta_p) n_b . \tag{22}$$

Since $0 \le \delta_p < 1$, $n_b > 0$, from (22) it follows

$$n_2^d \ge n_1^d . \tag{23}$$

Let the strength conditions (17) be satisfied for the factor n_b, i.e. let

$$n_1^p \le (1-\delta_p) n_b, \quad (1+\delta_p) n_b \le n_2^p .$$

Then for the factors n_1^d, n_2^d taking into account (23) there are the following inequalities

$$n_1^p \le n_1^d \le n_2^d \le n_2^p . \tag{24}$$

Comparing (20), (22) taking into account (19), there are the following inequalities

$$n_1^d \le n_1^r, \quad n_2^r \le n_2^d .$$

It follows from here that taking into account that according to (19) $n_1^r \le n_2^r$, we get

$$n_1^d \le n_1^r \le n_2^r \le n_2^d . \tag{25}$$

Then by reason of (24), (25) there are the following inequalities

$$n_1^p \le n_1^r \le n_2^r \le n_2^p . \tag{26}$$

From (26) taking into account (21) the fulfillment of the specified strength conditions (14) for the safety factor n_b^0 follows. The constraints on the parameter δ_p are found from the assumption of the existence of strength conditions (17), i. e. let

$$\frac{n_1^p}{1-\delta_p} \le \frac{n_2^p}{1+\delta_p} . \tag{27}$$

whence it follows

$$\delta_p \le C_p = \frac{\Delta n}{n_1^p + n_2^p} . \tag{28}$$

Note that as $n_2^p > n_1^p \ge 1$, then from (28) it follows $0 < C_p < 1$. If $\delta_p = C_p$, then the range for varying the values of the factor n_b equals to zero, i.e., in this case $n_b = (n_1^p + n_2^p)/2$ what is difficult to perform in practice at the given n_1^p, n_2^p. Thus, at $\delta_p < C_p$ it is possible to perform the given strength conditions (14) for the factor n_b^0 with the application of the adjusted strength conditions (17) and the approximate solution that generates for the stress σ_b such an error δ that $|\delta| \le \delta_p$. The theorem 2 is proved.

Corrected equivalent strength conditions taking into account the error of stresses. In practice, to solve the problems of the theory of elasticity formulated for three-dimensional composite structures, numerical methods are used, for example, FEM [10–12], which generate solutions with a small error. In this regard, it becomes necessary to take into account the error of solutions in equivalent strength conditions. In [16], equivalent strength conditions are considered without taking into account the error of approximate solutions. Using the results of [4], and based on Theorems 1 and 2, we formulate the adjusted equivalent strength conditions that take into account the error of the solutions. The adjusted equivalent strength conditions are reflected in the following theorem, which uses the notation introduced in Theorems 1, 2.

Theorem 3. Let the equivalent strength conditions of the form be determined for the safety factor of an elastic isotropic homogeneous structure V^b

$$n_1^p \le n_b^0 \le n_2^p\,, \tag{29}$$

where n_1^p, n_2^p are given (i. e. the parameter p is defined, see the theorem 1); n_b^0 – the safety factor of the structure V^b corresponding to the exact solution to three-dimensional elasticity problem (formulated for the structure V^b). Let for the structure V^b the maximum equivalent stress σ_b (corresponding to the approximate solution to the elasticity problem) be defined. Let

$$|\delta| \le \delta_p < C_p = \frac{\Delta n}{n_1^p + n_2^p}\,, \tag{30}$$

where $\Delta n = |n_2^p - n_1^p|$, δ – relative error for the stress σ_b, i. e. $\delta = (\sigma_b - \sigma_b^0)/\sigma_b^0$, where σ_b^0 – maximum equivalent stress of the structure V^b corresponding to the exact solution to the elasticity problem, δ_p – the estimate for the error δ.

Let the safety factor n_b of the structure corresponding to the approximate solution satisfy the adjusted equivalent strength conditions of the form

$$\frac{n_1^p}{1-\delta_p} \le n_b \le \frac{n_2^p}{1+\delta_p}\,. \tag{31}$$

where $n_b = \sigma_T / \sigma_b$, σ_T – yield stress.

Then, the safety factor n_b^0 of the structure corresponding to the exact solution satisfies the equivalent strength conditions (29), where $n_b^0 = \sigma_T / \sigma_b^0$.

The proof of the Theorem 3 is similar to the proof of the Theorem 2. Note that δ_p can be considered as the maximum error for the maximum equivalent structural stress. The relations (31) can be represented as

$$n_1^p(1+\varepsilon_1) \le n_b \le n_2^p(1-\varepsilon_2)\,,$$

or taking into account (13) we have

$$pn_1(1+\varepsilon_1) \le n_b \le pn_2(1-\varepsilon_2)\,, \tag{32}$$

where the values ε_1, ε_2 are defined with the help of the error δ_p of the stress σ_b according to the formulae

$$\varepsilon_1 = \frac{1}{1-\delta_p} - 1, \quad \varepsilon_2 = 1 - \frac{1}{1+\delta_p}\,, \quad 0 \le \delta_p < 1\,, \tag{33}$$

p – equivalence coefficient.

The fundamental principles of the method of equivalent strength conditions. Let us consider a cantilever composite beam V_0 having a regular structure (which is located in the Cartesian coordinate system $Oxyz$) with the length $L = 1536h = 600$, square section that has dimensions $H \times H$, where $H = 128h = 50$, fig. 1. The beam V_0 consists of plastic materials and has static loading $q_z(x, y, z)$.

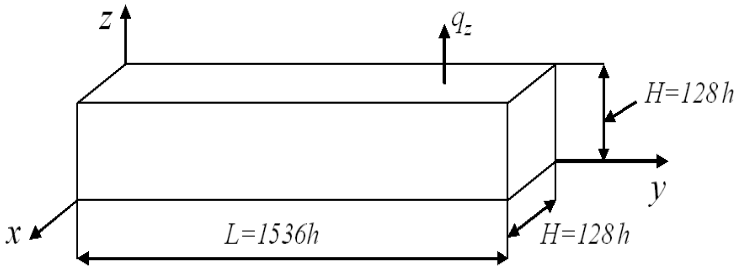

Fig. 1. The characteristic sizes of the beam V_0

Рис. 1. Характерные размеры балки V_0

The regular cell G_0 of a composite beam V_0 has the dimensions $8h \times 8h \times 8h$ in which longitudinal fibers with the cross section $h \times h$ are located, fig. 2, the cross sections of the fibers are shaded, 16 fibers. Thus, the beam is reinforced with longitudinal fibers – the section $h \times h$, the distance between the fibers is h. The fibers are isotropic homogeneous bodies and have the same elastic moduli.

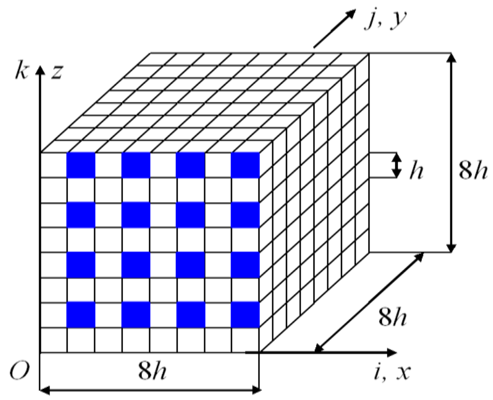

Fig. 2. Regular cell G_0

Рис. 2. Регулярная ячейка G_0

It is believed [17], if the fiber thickness is less than 0.5 mm, then such fibers form a micro-inhomogeneous structure. Let $L = 600$ мм, $H = 50$ мм, then, $h = 0{,}3906$ mm, i. e., the beam V_0 with the dimensions $5\,\text{см} \times 60\,\text{см} \times 5\,\text{см}$ has a microinhomogeneous regular fibrous structure.

For the safety factor n_0 of the composite beam V_0, the conditions of strength of the form (5) are specified. It is required to determine the safety factor n_0 of the given beam, i.e., check whether the beam satisfies the specified strength conditions. To solve this problem, we use the method of equivalent strength conditions, the fundamental principles of which will be considered (for simplicity, without losing the commonality of views) using an example of the beam V_0 with a microinhomogeneous regular structure. The basic regular partition $\mathbf{R}_0$ of the beam V_0 consists of (basic) single-grid FE (1gFE) V_j^h of the 1st order cube shapes with the side h [8] in which three-dimensional strain-stress state is realized. The partition $\mathbf{R}_0$ takes into account the microinhomogeneous structure of the beam, generates a uniform fine (base) grid with the step h of the dimension $129 \times 1537 \times 129$ and a discrete model with a total number of unknown nodal FEMs $N_0 = 76681728$, the width of the ribbon of FEM system of equations (CS) equals $b_0 = 50316$. The implementation of FEM for the base model $\mathbf{R}_0$ (more than 76 million nodal variables) requires large computer resources. The construction of a sequence of solutions is associated with the application of the grinding procedure for the basic partition, which is complex and difficult to implement for the composite structure of the beam V_0, since each grinding step leads to a sharp increase in the dimension of the discrete problem. Note that the step of the basic regular splitting $\mathbf{R}_0$ of the composite beam V_0 cannot be larger than h, since the fiber cross section has the dimensions $h \times h$.

According to MESC, we introduce an isotropic homogeneous beam V^b such that the beams V^b, V_0 have the same shape, dimensions, specified fastening and loading, but differ in elastic moduli. The elastic moduli of the beam V^b are equal to the elastic moduli of the fiber of the beam V_0. The implementation of MESC reduces to constructing the adjusted equivalent strength conditions (32) and to determining safety factor n_b of the body V^b, i. e., to determining the equivalence coefficient p for the beam V_0 and to determining the maximum equivalent stress σ_b for the body V^b using FEM with the error δ_p. The coefficient p is determined by the formula $p = \sigma_0 / \sigma_b$, where σ_0, σ_b are the maximum equivalent stresses of the bodies V_0, V^b, respectively. Note that finding the stress σ_0 by FEM (using single-grid FE cube shapes with side h [8]) for the beam V_0, taking into account its microinhomogeneous structure, requires large computer resources.

The following procedure is proposed to find the equivalence coefficient p and stress σ_b. For the isotropic homogeneous body V^b, we construct a sequence of basic regular partitions (discrete models) $\{V_n^0\}_{n=1}^N$ consisting of basic 1gFE $V_j^{(n)}$ of the first order of the shape of a cube with the side h_n. The discrete model V_n^0 has a dimension $n_1^{(n)} \times n_2^{(n)} \times n_3^{(n)}$, where

$$n_1^{(n)} = 8n+1,\ n_2^{(n)} = 96n+1,\ n_3^{(n)} = 8n+1,\ \ n = 1,..., N. \qquad (34)$$

The base grid steps on the axis Ox, Oy, Oz are $h_x^{(n)} = H/(8n)$, $h_y^{(n)} = L/(96n)$, $h_z^{(n)} = H/(8n)$, as $L = 12H$, то $h_n = h_x^{(n)} = h_y^{(n)} = h_z^{(n)}$ and $h_n > h$, $n = 1,..., N-1$. It should be noted that at $n = N$ we get $h_N = h$ (for the law of splitting (34) at $N = 16$ we have $h_{16} = h$), i. e. at $n = 16$ of the dimensions of an isotropic homogeneous discrete model V_{16}^0 and basic partition $\mathbf{R}_0$ of the composite beam V_0 are equal. It is important to note the following. The grinding law for partitions is specified so that each partition V_n^0 consists of a finite number of areas G_n^b of the same shape and size that the area G_n^b and the area of the regular cell (Fig. 2) have the same shape, but differ in characteristic sizes. For a given grinding law (34), the area G_n^b has dimensions $8h_n \times 8h_n \times 8h_n$. The area G_n^b differs from the area of the regular cell G_0 (in dimensions $8h \times 8h \times 8h$, Fig. 2) by the characteristic dimensions of the form $h_n = \beta_n h$, where $\beta_n > 1$. At $n \to 16$ we have $\beta_n \to 1$, at $n = 16$ we get $\beta_{16} = 1$

We introduce the area G_n^0 whose shape and characteristic dimensions coincide with the area G_n^b, $n = 1,..., N$. In this case, the area G_n^0 has a composite structure, which apparently coincides with the structure of a regular cell G_0, i.e., the area G_n^0 has the same number of fibers (with a square section in size $h_n \times h_n$) and the same mutual arrangement as in the cell G_0 (16 longitudinal fibers, fig. . 2). The fibers in the areas G_n^0, G_0 have the same moduli of elasticity. The areas G_n^0, G_0, in fact, differ only in scale, that is, it can be formally written $G_n^0 = \beta_n G_0$, where β_n is the scale factor, $\beta_n > 1$, $n = 1,..., N-1$. At $n = N$ we get $\beta_N = 1$, i. e. $G_N^0 = G_0$. For the grinding law (34), at $n = 16$ we have $\beta_{16} = 1$, i.e. $G_{16}^0 = G_0$. Note that the inhomogeneous (fibrous) structure is taken into account in the area G_n^0.

In the discrete model V_n^0, we replace all homogeneous isotropic areas G_n^b with the composite areas G_n^0. As a result, on the basis of an isotropic homogeneous model V_n^0, we get a composite (base) discrete model, which we denote by R_n^0 (in which the inhomogeneous structure is taken into account). Thus, at $n = 16$ the composite discrete model R_{16}^0 coincides with the basic model $\mathbf{R}_0$ of a composite beam V_0, i.e., we have $R_{16}^0 = \mathbf{R}_0$. The discrete models V_n^0, R_n^0 have the same shape, characteristic dimensions, the same fastening and loading, but differ in elasticity moduli. According to (34), the dimensions of the models V_n^0, R_n^0 increase sharply with increasing n. To lower the dimension of discrete models, multigrid finite elements are effectively applied [10, 11, 13, 14]. Using m- grid FEs in discrete basic models V_n^0, R_n^0 we get $m-$ grid discrete models V_n^b, R_n respectively, which have the same shape, characteristic dimensions, the same fastening and loading like the beam V_0, but they differ in elasticity moduli of the basic models V_n^0, R_n^0. The procedure for determining the equivalence coefficient is as follows. For the discrete models V_n^b, R_n, we determine respectively the maximum equivalent stresses σ_n^b, σ_n, with the help of which we find the coefficient $p_n = \sigma_n / \sigma_n^b$, $n = 1,...,N$. We have $p_n \to p$ at $n \to N$. Let $\delta_n = | p_n - p_{n-1} | / p_n$ be a small value, then we have $p = p_n$. Let the sequence of solutions $\{\sigma_n^b\}_{n=1}^{12}$ be constructed that quickly converges to an exact solution and let $\delta_n^\sigma = | \sigma_n^b - \sigma_{n-1}^b | / \sigma_n^b$ be a small value. Then we consider that σ_n^b is a maximum equivalent stress of an isotropic homogeneous body V^b (found with the error δ_p). Plugging the obtained p, δ_p and the given factors n_1, n_2 in (32), we determine the adjusted equivalent strength conditions for the composite beam V_0. The safety factor n_b for the body V^b is determined by the formula $n_b = \sigma_T / \sigma_n^b$, where σ_T is the yield strength of the fiber. If the found factor n_b satisfies the obtained adjusted equivalent strength conditions of the form (32), then the safety factor n_0 of the composite beam V_0 (fig. 1) satisfies the specified strength conditions of the form (5).

The results of numerical experiments. Let us consider the model problem of calculating the strength of a cantilever beam V_0 with a microinhomogeneous fibrous regular structure with the dimensions $128h \times 1536h \times 128h$, h - little, given, Fig. 1. The beam V_0 consists of plastic materials, has a square section with the dimensions $H \times H$, where $H = 128h$. The regular cell of the microinhomogeneous beam structure V_0 with the $8h \times 8h \times 8h$ has 16 identical longitudinal fibers with a cross section $h \times h$, Fig. 2, i. e. the beam is reinforced with isotropic homogeneous longitudinal fibers - section $h \times h$, the distance between the fibers is h. At $y = 0$: $u = v = w = 0$, i.e. in the plane xOz, the beam V_0 is fastened. For the safety factor n_0 of the beam V_0 the strength conditions of the form are given

$$1{,}3 \le n_0 \le 3{,}2. \tag{35}$$

For the beam V_0 we use the following basic data:

$$h = 0{,}3906;\ \sigma_T = 5;\ E_v = 10,\ E_c = 1,\ \nu_c = \nu_v = 0{,}3,\ \ q_z = 0{,}0018, \tag{36}$$

where E_c, E_v (ν_c, ν_v) - Young's modulus (Poisson's ratio) of a binding material and fibers, respectively, σ_T - fiber yield strength, loading q_z acts on the surface $z = H$, $0{,}5L \le y \le L$, fig. 1.

To calculate the beam V_0 we use the equivalent strength method using multigrid finite elements. In the calculations, we use homogeneous and composite Lagrangian three-grid FEs (3gFE) having the shape of a rectangular parallelepiped. We will consider the fundamental principles for constructing 3gFE using the example of a composite 3sFE $V_\alpha^{(3)}$ having the shape of a rectangular parallelepiped with the dimensions $8h \times 16h \times 8h$ [10, 15]. 3gFE $V_\alpha^{(3)}$ is located in the local Cartesian coordinate system $Oxyz$, which contains two regular cells G_0 with the dimensions $8h \times 8h \times 8h$ of the composite beam V_0. First, we consider the procedure for constructing a composite Lagrangian two-grid FE (2sFE) $V_d^{(2)}$ with the dimensions $8h \times 8h \times 8h$ that contains one regular cell G_0. In the procedure we use a uniform fine mesh h_d with the step h and the dimensions $9 \times 9 \times 9$ the course mesh H_d nested in the fine mesh, $H_d \subset h_d$. Fig. 3 shows a fine mesh h_d and a course mesh H_d having 125 nodes, which are marked with dots. The fine mesh h_d is generated by the basic R_d 2gFE $V_d^{(2)}$, which consists of 1gFE V_j^h of the 1st order cube shape with the side h (in which three-dimensional strain-stress state is realized, $j = 1,...,M$, M is the total number of 1gFE, $M = 512$) and which takes into account the microinhomogeneous structure of 2 gFE $V_d^{(2)}$. The fibers are parallel to the axis Oy, the cross sections of the fibers in the plane are shaded, 16 fibers, Fig. 3.

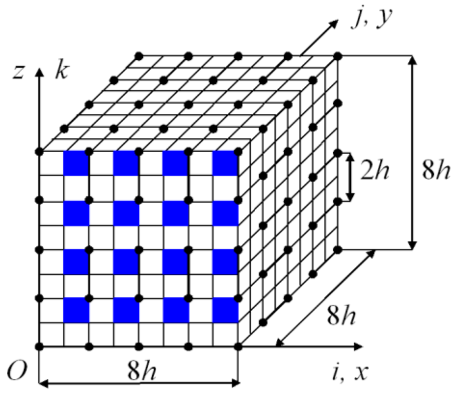

Fig. 3. Small and large mesh 2gFE $V_d^{(2)}$

Рис. 3. Мелкая и крупная сетки 2сКЭ $V_d^{(2)}$

The full potential energy P_d of the base partition R_d 2gFE $V_d^{(2)}$ is presented [5; 8]

$$\Pi_d = \sum_{j=1}^{512} \left(\frac{1}{2}\mathbf{q}_j^T [K_j^h] \mathbf{q}_j - \mathbf{q}_j^T \mathbf{P}_j\right), \tag{37}$$

where $[K_j^h]$ – stiffness matrix, $\mathbf{P}_j$, $\mathbf{q}_j$ – the vectors of nodal forces and displacements of 1gFE V_j^h of the base partition 2gFE, T – transposition.

Using Lagrange polynomials [5] on the large mesh H_d we define approximating displacement functions u_2, v_2, w_2 for 2gFE $V_d^{(2)}$, which are written in the form

$$u_2 = \sum_{i=1}^{5}\sum_{j=1}^{5}\sum_{k=1}^{5} N_{ijk} u_{ijk}, \quad v_2 = \sum_{i=1}^{5}\sum_{j=1}^{5}\sum_{k=1}^{5} N_{ijk} v_{ijk}, \quad w_2 = \sum_{i=1}^{5}\sum_{j=1}^{5}\sum_{k=1}^{5} N_{ijk} w_{ijk}, \tag{38}$$

where u_{ijk}, v_{ijk}, w_{ijk} – displacement values u, v, w in the node i, j, k of the mesh H_d; i, j, k – coordinates of an integer coordinate system ijk, introduced for the mesh nodes H_d (fig. 3); $N_{ijk} = N_{ijk}(x, y, z)$ – base function of the node i, j, k of the mesh H_d, $i, j, k = 1,...,5$, $N_{ijk} = L_i(x) L_j(y) L_k(z)$, where

$$L_i(x)=\prod_{\alpha=1,\alpha\neq i}^{5}\frac{x-x_\alpha}{x_i-x_\alpha},\quad L_j(y)=\prod_{\alpha=1,\alpha\neq j}^{5}\frac{y-y_\alpha}{y_j-y_\alpha},\quad L_k(z)=\prod_{\alpha=1,\alpha\neq k}^{5}\frac{z-z_\alpha}{z_k-z_\alpha}, \tag{39}$$

x_i, y_j, z_k – coordinates of the node i, j, k of the mesh H_d in the coordination systemт $Oxyz$, fig. 3.

Let us denote: $N_\beta = N_{ijk}$, $u_\beta = u_{ijk}$, $v_\beta = v_{ijk}$, $w_\beta = w_{ijk}$, where $i, j, k = 1,...,5$, $\beta = 1,...,125$. Then the equations (38) take the form

$$u_2=\sum_{\beta=1}^{125}N_\beta u_\beta,\quad v_2=\sum_{\beta=1}^{125}N_\beta v_\beta,\quad w_2=\sum_{\beta=1}^{125}N_\beta w_\beta. \tag{40}$$

We denote by $\mathbf{q}_d=\{u_1,...,u_{125}, v_1,...,v_{125}, w_1,...,w_{125}\}^T$ nodal displacement vector of the mesh H_d, i. e. nodal unknown vector 2gFE $V_d^{(2)}$. Using (40), the components of the vector $\mathbf{q}_j$ of the nodal variables 1gFE V_j^h are expressed in terms of the components of the vector $\mathbf{q}_d$, as a result we get the equation

$$\mathbf{q}_j=[A_j^2]\,\mathbf{q}_d, \tag{41}$$

where $[A_j^2]$ – rectangular matrix, $j = 1,...,512$.

Substituting (41) into equation (37), from the condition $\partial\Pi_d/\partial\mathbf{q}_d = 0$ we get $[K_d]\,\mathbf{q}_d = \mathbf{F}_d$, where

$$[K_d]=\sum_{j=1}^{512}[A_j^2]^T[K_j^h][A_j^2],\qquad \mathbf{F}_d=\sum_{j=1}^{512}[A_j^2]^T\,\mathbf{P}_j, \tag{42}$$

where $[K_d]$ – stiffness matrix (the dimension – 375×375), $\mathbf{F}_d$ – nodal force vector (the dimension – 375) 2gFE $V_d^{(2)}$.

Let us consider the construction of the Lagrangian three-grid FE (3gFE) $V_\alpha^{(3)}$, using two 2gFE $V_d^{(2)}$. The small mesh h_α and the large one H_α of 3gFE $V_\alpha^{(3)}$ are shown in fig. 4, the nodes of the mesh H_α are marked with dots, 12 nodes. The nodes of the small mesh h_α are the nodes of the large meshes H_d of two 2gFE $V_d^{(2)}$, $d = 1, 2$.

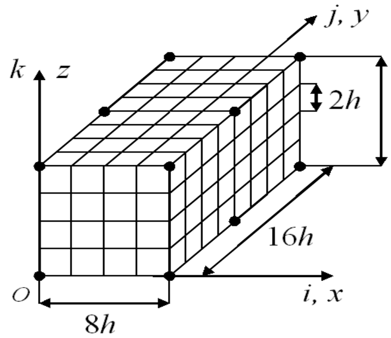

Fig. 4. Small h_α and large H_α mesh 3gFE $V_\alpha^{(3)}$

Рис. 4. Мелкая h_α и крупная H_α сетки 3сКЭ $V_\alpha^{(3)}$

The full potential energy P_α 3gFE $V_\alpha^{(3)}$ can be presented in the form

$$P_\alpha=\sum_{d=1}^{2}\left(\frac{1}{2}\mathbf{q}_d^T[K_d]\mathbf{q}_d-\mathbf{q}_d^T\mathbf{F}_d\right), \tag{43}$$

where $[K_d]$, $\mathbf{F}_d$, $\mathbf{q}_d$ – stiffness matrix, vectors of nodal forces and displacements 2gFE $V_d^{(2)}, d = 1, 2$.

Using Lagrange polynomials on the large mesh H_α we determine the approximating functions of displacements u_3, v_3, w_3 for 3gFE $V_\alpha^{(3)}$, that are written in the form

$$u_3 = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{2} N_{ijk} u_{ijk}\ , \quad v_3 = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{2} N_{ijk} v_{ijk}\ , \quad w_3 = \sum_{i=1}^{2}\sum_{j=1}^{3}\sum_{k=1}^{2} N_{ijk} w_{ijk}\ , \tag{44}$$

where $u_{ijk},\ v_{ijk},\ w_{ijk}$ – displacement values of u, v, w in the node i, j, k of the mesh H_α; i, j, k – coordinates of an integer coordinate system ijk, introduced for the mesh H_α (fig. 4); $N_{ijk} = N_{ijk}(x, y, z)$ – base function of the node i, j, k of the mesh H_α, $i,k = 1,2$, $j = 1,2,3$, $N_{ijk} = L_i(x) L_j(y) L_k(z)$, where

$$L_i(x) = \prod_{\alpha=1,\alpha\neq i}^{2} \frac{x - x_\alpha}{x_i - x_\alpha}\ , \quad L_j(y) = \prod_{\alpha=1,\alpha\neq j}^{3} \frac{y - y_\alpha}{y_j - y_\alpha}\ , \quad L_k(z) = \prod_{\alpha=1,\alpha\neq k}^{2} \frac{z - z_\alpha}{z_k - z_\alpha}\ , \tag{45}$$

x_i, y_j, z_k – node coordinates i, j, k of the mesh H_α in the coordinate system $Oxyz$, fig. 4.

let us denote $N_\beta = N_{ijk}$, $u_\beta = u_{ijk}$, $v_\beta = v_{ijk}$, $w_\beta = w_{ijk}$, where $i,k = 1, 2$, $j = 1, 2, 3$, $\beta = 1,...,12$. Then the equations (44) take the form

$$u_3 = \sum_{\beta=1}^{12} N_\beta u_\beta\ , \quad v_3 = \sum_{\beta=1}^{12} N_\beta v_\beta\ , \quad w_3 = \sum_{\beta=1}^{12} N_\beta w_\beta\ . \tag{46}$$

We denote by $\mathbf{q}_\alpha = \{u_1,...,u_{12}, v_1,...,v_{12}, w_1,...,w_{12}\}^T$ the vector of nodal displacements of the large mesh H_α, i.e., the vector of nodal variables 3gFE $V_\alpha^{(3)}$. Using (46), the vector components $\mathbf{q}_d$ of nodal variables 2gFE $V_d^{(2)}$ are expressed by the vector components $\mathbf{q}_\alpha$, as a result we get

$$\mathbf{q}_d = [A_d^3]\,\mathbf{q}_\alpha\,, \tag{47}$$

where $[A_d^3]$ – rectangular matrix, $d = 1, 2$.

Substituting (47) into the expression (43), from the condition $\partial P_\alpha / \partial \mathbf{q}_\alpha = 0$ we get $[K_\alpha]\,\mathbf{q}_\alpha = \mathbf{F}_\alpha$, where

$$[K_\alpha] = \sum_{d=1}^{2} [A_d^3]^T [K_d][A_d^3]\,, \quad \mathbf{F}_\alpha = \sum_{d=1}^{2} [A_d^3]^T\, \mathbf{F}_d\ , \tag{48}$$

where $[K_\alpha]$ – stiffness matrix (the dimension- 36×36), $\mathbf{F}_\alpha$ - nodal force vector (the dimension – 36) 3gFE $V_\alpha^{(3)}$.

<u>*Comment 2.*</u> The solution built for the large mesh H_α 3gFE $V_\alpha^{(3)}$, using the formula (47) is projected onto the small mesh h_α 3gFE $V_\alpha^{(3)}$. Then, using the formula (41), we determine the nodal displacements of the basic partitions 2gFE $V_d^{(2)}$, which makes it possible to calculate stresses in any 1gFE V_j^h of base partition R_d 2gFE $V_d^{(2)}$, $d = 1, 2$.

<u>*Comment 3.*</u> By virtue of (41), the dimension of the vector $\mathbf{q}_d$ (i.e. the dimension of 2gFE $V_d^{(2)}$) does not depend on the total number M of 1gFE V_j^h, i.e. on the dimension of the partition R_d. Therefore, to take into account the microinhomogeneous structure in 2gFE, one can use arbitrarily small basic partitions consisting of 1gFE V_j^h. In this case in 2gFE $V_d^{(2)}$ (consequently, in 3gFE $V_\alpha^{(3)}$) the three-dimensional strain-stress state is described arbitrarily accurately (without introducing additional simplifying hypotheses).

Using the procedures described in [14], 2gFE are designed to calculate composite shells of revolution, rings of complex shape and shafts that have central circular holes, composite and uniform cylindrical shells, plates and beams of complex shape, which are widely used in practice. The procedure for constructing homogeneous multigrid finite elements is similar to the procedure for constructing composite multigrid finite elements.

For the composite beam V_0 we define an isotropic homogeneous body V^b (the beam V^b). The bodies V_0, V^b have the same shape, size, fastening and loading, the elastic moduli of the body V^b are equal to the elastic moduli of the fiber. Using the law of splitting (34), we construct, according to the procedure described above, three-mesh discrete models V_n^b, R_n, consisting respectively of isotropic homogeneous and composite 3gFE of the type $V_\alpha^{(3)}$ with the dimensions $8h_n \times 16h_n \times 8h_n$, $n=\overline{1,12}$. For the discrete isotropic homogeneous model V_n^b we find the solutions w_n^b, σ_n^b, where w_n^b, σ_n^b - maximum displacement and equivalent voltage of the discrete model V_n^b, $n=1,...,12$. Equivalent stresses are determined by the 4th theory of strength. The calculation results are presented in Table 1. The analysis of the calculation results shows a fast uniformly monotonic convergence of approximate solutions (w_n^b, σ_n^b) to the exact solution. The stresses $\sigma_{11}^b = 0{,}665$, $\sigma_{12}^b = 0{,}686$, differ by $\delta = 3{,}061\%$. The test calculations show that in this case the stress σ_{12}^b is found with the error $10\% \div 15\%$. Let $\delta_p = 0{,}15$. The condition (30) for δ_p is satisfied. Taking into account relations (13) and (35) in (30), we have $\delta_p = 0{,}15 < C_p = 0{,}42$. According to (33) at $\delta_p = 0{,}15$ we get $\varepsilon_1 = 0{,}176$, $\varepsilon_2 = 0{,}131$. The adjusted equivalent strength conditions (32) for $\varepsilon_1 = 0{,}176$, $\varepsilon_2 = 0{,}131$ have the form

$$1{,}176\, pn_1 \le n_b \le 0{,}869\, pn_2 , \tag{49}$$

where n_b - safety factor of the body V^b, determined by FEM.

Таблица 1

The results of calculations of the beam V^b

Результаты расчетов балки V^b

V_n^b	w_n^b	σ_n^b	V_n^b	w_n^b	σ_n^b
V_1^b	204,851	0,377	V_7^b	238,033	0,569
V_2^b	228,503	0,489	V_8^b	238,263	0,595
V_3^b	234,023	0,524	V_9^b	238,422	0,620
V_4^b	236,109	0,537	V_{10}^b	238,545	0,643
V_5^b	237,119	0,543	V_{11}^b	238,630	0,665
V_6^b	237,683	0,547	V_{12}^b	238,697	0,686

Note that the three-grid discrete model V_{12}^b, consisting of Lagrangian 3gFE of the type $V_\alpha^{(3)}$ ($\alpha = 1,...,32768$) with the dimension $8h_{12} \times 16h_{12} \times 8h_{12}$, has $N_{12}^b = 73008$ of nodal variables of FEM, the width of the tape the control system of FEM is $b_{12} = 1059$. The implementation of FEM for the discrete model V_{12}^b requires $k = \dfrac{N_0 \times b_0}{N_{12}^b \times b_{12}} = \dfrac{76681728 \times 50316}{73008 \times 1059} = 49903{,}566$ times less computer memory than for the base model $\mathbf{R}_0$ of the beam V_0, which shows the high efficiency of the application of Lagrangian 3gFE of the type $V_\alpha^{(3)}$ in the calculations. The equivalence coefficient p for the composite beam V_0 is determined using the procedure described above. The discrete models

V_n^b, R_n, $n=9,11,12$ are constructed using 3gFE of the type $V_\alpha^{(3)}$ based on basic regular partitions, respectively, with the dimensions: $73\times865\times73$, $89\times1057\times89$ и $97\times1153\times97$. The equivalence coefficients p_n are found by the formula $p_n=\sigma_n/\sigma_n^b$, where σ_n, σ_n^b – maximum equivalent stresses of the models R_n, V_n^b, $n=9,11,12$ respectively. As a result of the calculations, we get: $p_9=3{,}002$, $p_{11}=3{,}000$, $p_{12}=2{,}999$. The relative errors for the found coefficients p_9, p_{11}, p_{12} are

$$\delta_1(\%)=100\%\times|p_{11}-p_9|/p_{11}=100\%\times|3{,}002-3{,}000|/3{,}000=0{,}066\%,$$
$$\delta_2(\%)=100\%\times|p_{22}-p_{11}|/p_{12}=100\%\times|3{,}000-2{,}999|/2{,}999=0{,}033\%.$$

As $p_9>p_{11}>p_{12}$ and δ_2 is the smallest value, then we assume that the equivalence coefficient is $p=p_{12}=2{,}999$. Plugging into (49) $p=2{,}999$, $n_1=1{,}3$, $n_2=3{,}2$, we get

$$4{,}584\le n_b\le 8{,}339. \quad (50)$$

The safety factor of the homogeneous body V^b is $n_b=\sigma_T/\sigma_{12}^b=5/0{,}686=7{,}288$, which satisfies the adjusted equivalent strength conditions (50). This means that the safety factor n_0 of the composite beam V_0 satisfies the specified strength conditions (35).

Let us perform verification calculations. Based on the underlying partition $\mathbf{R}_0$ of the beam V_0 using 3gFE $V_\alpha^{(3)}$ we build three-grid discrete models: composite R_{16} and isotropic homogeneous R_{16}^b corresponding to the law of splitting (34) at $n=16$. We consider that the stresses $\sigma_{16}=2{,}279$, $\sigma_{16}^b=0{,}762$ correspond to exact solutions, i.e. $\sigma_0=\sigma_{16}$, $\sigma_b=\sigma_{16}^b$. Then the safety factor for the composite body V_0 is $n_0=\sigma_T/\sigma_0=5/2{,}279=2{,}194$, i.e. $n_0=2{,}194$ satisfies given strength conditions (35), which confirms a similar conclusion obtained using MESC.

The equivalence coefficient p_0 (corresponding to the exact solutions) for the beam V_0 is $p_0=\sigma_0/\sigma_b=2{,}279/0{,}762=2{,}990$. It should be noted that the coefficients $p=2{,}999$ and $p_0=2{,}990$ differ by 0,301 %, i.e. as a matter of fact, you can take $p_0=p$.

The dimension of the base discrete model V_{12}^0 (whose mesh at $n=12$ has the dimension $97\times1153\times97$, see formulae (34)) equals 32517504, the width of the tape of the control system of FEM is 28524. The number of nodal variables of FEM of the three-grid discrete model V_{12}^b equals 73008, the width of the tape of the control system of FEM is 1059. The implementation of FEM for a homogeneous isotropic three-grid discrete model V_{12}^b requires $k_2=\dfrac{32517504\times28524}{73008\times1059}=11996{,}685$ times less computer memory than for the base model V_{12}^0, consisting of the known 1gFE of the cube shape with the side h_{12}.

Conclusion. The method of equivalent strength conditions is proposed for calculating the static strength of structures (plates, beams, shells) with an inhomogeneous, microinhomogeneous regular structure under specified strength conditions. The implementation of the method is reduced to calculating the strength of isotropic homogeneous bodies with the use of equivalent strength conditions built on the basis of given ones. When calculating homogeneous bodies according to FEM, multigrid finite elements are used, which generate discrete models of small dimension and solutions with a small error. The implementation of the proposed method requires small computer resources.

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